The Independence of the Continuum Hypothesis in Zermelo-Fraenkel Set Theory

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Introduction

The creation of modern set theory was largely initiated by Georg Cantor and his work in developing a rigorous and precise mathematical notion of infinity. Notably, Cantor showed that the cardinality of the set of natural numbers is strictly less than the cardinality of the set of real numbers. This ultimately led to an investigation on the organization of these types of infinities; more specifically, Cantor wanted to know whether there was an infinity between these two infinities. The Continuum Hypothesis (CH) was proposed by Cantor in 1878, which states that $2^{\aleph_0} = \aleph_1$, that is, the cardinality of the power set of integers is the first uncountable number. Generally, the intuition is that infinities are organized by applying the power set operation to one infinity to generate the next. However, despite all of his best attempts, Cantor was unable to prove this hypothesis, and it isn't until over forty years after his work that mathematics obtains an explanation on CH's status. Kurt Gödel shows that the Continuum Hypothesis is consistent with the axioms of set theory, and Paul Cohen shows that the negation of the hypothesis can also be added consistently; these two results demonstrate its independence, namely, that it is provably unprovable given the system generated by the axioms of set theory.

This paper will attempt to accomplish three things. First, I will outline the history of the Continuum Problem and its place in mathematics and philosophy, emphasizing the revolutionary role Cantor's mathematics and set theory have had on the characterization of the continuum and notion of the infinite. Then, I will provide a simplified summary of the independence proofs, as adapted from the original papers and subsequent published explanations. From this, I will offer reflections on the proof techniques and model theory used within the proof, and connect the independence proof to a broader conversation of philosophical ideas of truth in mathematics, discussing their semantic implications.

Cantor and the Study of infinity

Georg Cantor is credited for developing the notion of one-to-one correspondence to establish equivalence relationships between what he called sets–sets A and B are said to be of equal cardinality if and only if there exists a bijective function from one to the other.[1] Cantor defines countability as the ability to be put into a one-to-one correspondence with the natural numbers.[2] Take for example, the set of perfect squares $P = \{1, 4, 9, 16, \ldots, n^2, ...\}$. Cantor's definition of enumerability allows us to conclude that P is *just as big* as $\mathbb{N}$, even though it is by definition a subset of $\mathbb{N}$, and we can clearly find natural numbers which are missing from this set.[3]

Cantor proves in 1873 that the reals have the property of being uncountable; there exists no bijection between the real numbers and the integers. He originally uses what we would now call real analysis, but then provides a more intuitive *reductio ad absurdum* proof nineteen years later.[4] The proof essentially goes as follows:

Proof. Assume that the real numbers are countable, and consider an arbitrary enumera-

[1] Gödel, Kurt. "What Is Cantor's Continuum Problem?" The American Mathematical Monthly 54, no. 9 (1947): 515-25. doi:10.2307/2304666, 515.

[2] Finite sets are always countable, since they can be mapped to a finite subset of $\mathbb{N}$.

[3] The bijective function which establishes the one-to-one correspondence is $f : P \to \mathbb{N}; f(n) = n^2$.

[4] Ferreirós Domínguez José. Labyrinth of Thought : A History of Set Theory and Its Role in Modern Mathematics. Science Networks Historical Studies, V. 23. Basel, Switzerland: Birkhäuser Verlag, 1999, 181.

tion:

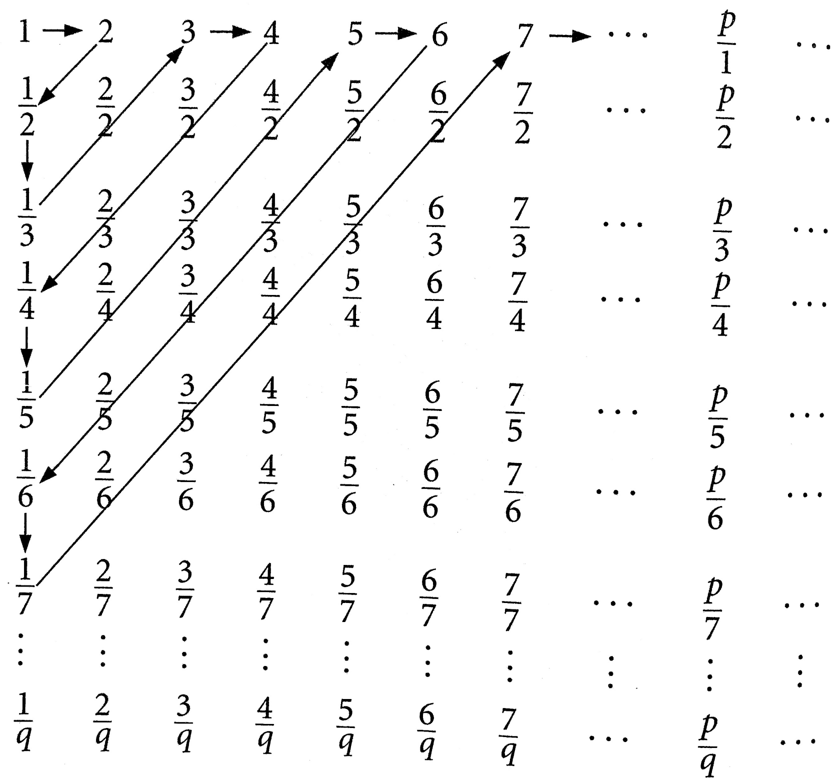
$$0 \longleftrightarrow 0.a_1a_2a_3a_4\dots$$
$$1 \longleftrightarrow 0.b_1b_2b_3b_4\dots$$
$$2 \longleftrightarrow 0.c_1c_2c_3c_4\dots$$
$$3 \longleftrightarrow 0.d_1d_2d_3d_4\dots$$
$$\dots$$

Consider the number $0.a_1b_2c_3d_4\dots$, that is, the decimal formed by the taking the diagonal entries of each number in the enumeration. To it, add the number $0.111111\dots$, and call this sum r. The number r will not be in the enumeration, because it differs from the n-th number in the numeration in the n-th place. Since the enumeration is arbitrary, we will always be able to find the corresponding r for any enumeration.[5] $\square$

With this proof, Cantor has shown the existence of at least two types of infinities–countable infinities equinumerous with $\mathbb{N}$, and an uncountable infinity, that of $\mathbb{R}$. Naturally, this begins an investigation into the cardinality of other mathematical sets, and this yields a bounty of surprising results, notably the enumerability of the rationals.

[5]Technically, we should really say that this is a proof that the real numbers in $(0,1)$ are uncountable, we can further prove that $(0,1)$ and $\mathbb{R}$ have the same cardinality.

Consider the rationals arranged in the following way:[6]



The rationals are listed in an infinite array according to their numerators (vertical columns) and denominators (horizontal columns) and it's easy to see that every rational number will appear in this array. Then, following the arrows gives an effective way of enumerating them, with the condition that we only count rational numbers in their most simplified form so that we account for them only once.

This proof reaffirms the power of Cantor's simple notion of one-to-one correspondence and explains why this notion is now inextricably linked to the study of the infinite–our intuitions of what determine "bigger sets" offer no guidance in navigating the world of infinities, so we need mathematically precise methods of comparing and counting. However, Cantor's interest grows in the direction of the organization and types of infinities, not just the division between countably and uncountably infinite sets. The key is found in Cantor's Theorem of Power Sets, a result he proves in 1892.[7] The power set $\wp(A)$ of

[6] Wallace, David Foster. Everything and More: A Compact History of Infinity. New York (NY),: W.W. Norton, 2010, 252.

[7] Ferreirós, José. "The Early Development of Set Theory." Stanford Encyclopedia of Philosophy. July

A is the collection of all the subsets of A, and Cantor shows that the cardinality of a set A is always strictly less than the cardinality of $\wp(A)$.[8] This is to say, that for every set, there exists a set that's bigger than it–even for ininite sets, we can always find larger infinities. This leads to the infamous hierarchy of aleph-numbers:

$$\aleph_0, \aleph_1, \aleph_2, \ldots$$

$\aleph_0$ is the cardinality of the integers, and $\aleph_1$ is the first uncountable number, and each $\aleph$ which follows denotes a larger infinity.[9] The aleph-hierarchy shows us that there are actually an infinite number of types of infinities, each one bigger than the previous.

Thus, the stage is set for Cantor to bridge his system of the infinite to the real number line. Cantor hasn't shown where c fits into his hierarchy, and one can, at this point, plausibly conjecture that the size of the continuum can be any $\aleph_\alpha$, or maybe doesn't fit into this hierarchy at all. Recall that Cantor's theorem only shows that for every set, there exists a set bigger than it (the power set), but whether the power set operation is how the alephs are organized, that is, whether $\wp(\aleph_0) = \aleph_1$, is not addressed by the theorem. Cantor conjectures that this is true, namely, that:

$$2^{\aleph_0} = \aleph_1$$

Another way to think about this problem is that the hypothesis claims there is no set with cardinality strictly greater than the integers but strictly less than the reals. This is known as the Continuum Problem, and despite all his efforts, Cantor is unable to prove this statement.[10] It's unclear whether this problem took a serious psychological toll on Cantor, but concurrent with the lack of answer to this problem is a series of mental breakdowns which leaves him visiting mental institutions and prevents him from devoting

01, 2016. Accessed March 12, 2019. https://plato.stanford.edu/entries/settheory-early/.

[8]Hunter, Geoffrey. Metalogic: An Introduction to the Metatheory of Standard First Order Logic. Berkeley: University of California Press, 1996, 24.

[9]Gödel, Kurt. "What Is Cantor's Continuum Problem?", 517.

[10]Wallace, 293.

his energy to his work.[11] The problem becomes such a central source of contention that Hilbert placed it as the first problem on his list presented at the International Congress of Mathematics.[12] However, it isn't until Gödel starts work on CH that the mathematics community receives any significant results on CH's status.

Independence Results

In order to properly place the Continuum Problem within the story of mathematics, it's important to understand the set theory which provides the foundation for the study of the infinite. Cantor's work is a large motivator for the move to make set theory rigorous and grounded; while it's shown to be a powerful way of describing mathematical objects and proofs, taken naively, set theory lends itself to paradoxes, as most notably shown by Russell and Cantor. In 1908, Ernst Zermelo proposes an axiomatic approach to set theory which disallows these paradoxes, and later work by Fraenkel, Skolem, and von Neumann modify the axioms to produce what is now referred to as the Zermelo-Fraenkel (ZF) set theory.[13] This system becomes the standard axiomatization and it's within this system that the independence results are conducted.

Gödel, 1938

Gödel publishes a paper in 1940 showing a model of the ZF axioms and CH.[14] A model is an interpretation of a formal system which makes all of its axioms and theorems true–thus, if Gödel can construct a model which satisfies the ZF axioms and the Continuum Hypothesis, it will show that the system $ZF + CH$ is consistent. Note that this is not equivalent to the project of deriving CH from the ZF axioms–the claim here isn't that

[11]Dauben, Joséph W. "Georg Cantor and the Origins of Transfinite Set Theory." Scientific American 248, no. 6 (1983): 122-31. http://www.jstor.org/stable/24968925, 129.

[12]Koellner, Peter. "The Continuum Hypothesis." Stanford Encyclopedia of Philosophy. May 22, 2013. Accessed March 12, 2019. https://plato.stanford.edu/entries/continuum-hypothesis/

[13]Bagaria, Joan. "Set Theory." October 08, 2014. Accessed March 11, 2019. https://plato.stanford.edu/archives/win2017/entries/set-theory/

[14]Ferreirós, 382.

CH is true. Gödel's proof uses a technique which is now known as creating an "inner model", a class of sets which has a restriction on which sets can be its members.

In order to understand Gödel's construction, we first begin by considering the von Neumann hierarchy:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \wp(V_\alpha) \\ V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha \\ V &= \bigcup_{\alpha} V_\alpha \end{aligned}$$

V is a class of sets which starts with the empty set and builds up using the power set as an operator, carrying over to the transfinite ordinals, and we can actually prove as a theorem that every set is in some V_α.[15] V is the union of all V_α's, so intuitively we can think of V as a universe containing all sets.[16] Gödel takes this building process and modifies the construction to include only definable sets, that is, each level of the hierarchy is constructed by subsets definable by formulas of ZF ranging over the previous level.[17] The structure of this universe mirrors the von Neumann hierarchy closely, with one notable difference:

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= def(L_\alpha) \\ L_\lambda &= \bigcup_{\alpha<\gamma} L_\alpha \\ L &= \bigcup_{a\in On} L_\alpha \end{aligned}$$

L is the union of the levels of the hierarchies, and we call it the "constructible hierarchy".

[15]Wolf, Robert S. A Tour through Mathematical Logic. The Carus Mathematical Monographs, No. 30. Washington, DC: Mathematical Association of America, 2005, 92

[16]This is an important, purposeful usage of the word "universe". V cannot be a set–the reader is invited to justify this to themselves.

[17]Wolf, 229

A set x is said to be constructible if $x \in L_\alpha$ for some α.[18]

It's hard to understand the explicit difference between V and L, other than a basic intuition that L denotes a more exclusive universe, because of the extra restriction of definability. For ordinals $\alpha \leq \omega$, the levels of the hierarchies look exactly the same–the power set operation is definable on all finite sets.[19]. However, $\wp(\omega)$ is uncountable in the von Neumann universe, but when we try to find $\wp(\omega)$ in $L_{\omega+1}$, we won't have enough defining power to express $\wp(\omega)$, so (the analogue of) $\wp(\omega)$ in L will still have to be countable.[20] Gödel's idea is to show that L is sufficiently similar to V so that all of the ZF axioms should still hold in L, or more precisely, that L is a model of the ZF axioms. When assuming that $V = L$, known as the Axiom of Constructibility, Gödel is able to show that every real number in L is constructed by a countable ordinal, or more formally, if $x \in L$ and $x \subset \omega$, then $x \in L_\alpha$ for some countable α.[21] After establishing this, we know that the number of countable ordinals is $\aleph_1$, so the cardinality of $\wp(\omega)$ in L is at most $\aleph_0 \times \aleph_1$ which is $\aleph_1$, and it follows that $2^{\aleph_0} = \aleph_1$.[22] Thus, Gödel effectively shows the existence of a model which has all the properties needed to confirm the consistency of the Continuum Hypothesis with ZF.[23] [A]

Paul Cohen, 1963

Paul Cohen's approach to the problem has a fundamentally similar intuition of discovering the right model, but yields a completely different result. Cohen's proof involves taking a standard transitive model of ZF and enlarging it by adding a set, and this is done in such a way as to preserve the ZF axioms, but violate the Continuum Hypothesis. With the benefit of hindsight, this seems like a obvious approach to the problem–if Gödel is able

[18] Ferreirós, 383.

[19] Devlin, Keith J. Fundamentals of Contemporary Set Theory. New York: Springer, 1979, 152

[20] Ibid., 153.

[21] Wolf, 234.

[22] Ibid., 234.

[23] Gödel actually does something more impressive, details of which we are skipping–he's able to show that the Axiom of Choice holds in this model, and then the Generalized Continuum Hypothesis holds as well. Thus, the full expression of the theorem is given: (Consis ZF) $\rightarrow$ (Consis ZF + AC + GCH) See (Cohen 1966).

to provide a model whose success is built around its "minimality", we simply reverse the direction and maximize the model so that the negation of the hypothesis holds. Cohen's genius was essentially inventing, from scratch, a proof technique called "forcing", a formal way of expressing *how* the models ought to be enlarged.

Cohen begins with a standard transitive countable minimal model M of ZF. In order to take this model and enlarge it, he adds an infinite set of integers a to it, arriving at a new model $M(a)$ which violates the Axiom of Constructibility but is still a model of ZF. What properties $M(a)$ has is largely dictated by how we define a.[24] This is the power of forcing–if one were to add a concretely defined set of integers, we could trivially show that this set would be found on a level of the set hierarchy.[25] Instead of adding these sets, Cohen forces them to be in the new model. For Cohen and the Continuum Hypothesis, this simply means specifying the conditions so that the cardinality of the power set of the integers is more than $\aleph_1$, that is, adding more than $\aleph_1$ new subsets of integers. In Cohen's forced model, $c \geq \aleph_\tau$, where $\tau \geq 2$. Thus, Cohen's forcing allows $2^{\aleph_0}$ to be greater than $\aleph_n, \aleph_\omega, \aleph_{\aleph_\omega}$, etc.[26]

These results, combined with Gödel's arguments, show the independence of the Continuum Hypothesis–it can be neither be proved nor disproved within ZF. Gödel has shown a model of (ZF + CH), and Cohen has shown a model of (ZF + $\sim$CH), so the Continuum Hypothesis can't be related to the ZF axioms by proof. It holds the status of being provably unprovable.[B]

Reflections on proof techniques

Forcing becomes a mathematical field of its own as an important proof technique particularly for establishing independence. The power of the technique at times seems paradoxical to the simple intuition or goal of forcing, that is, the notion of building or extending

[24] Ibid., 1092.

[25] Cohen, Paul. "The Discovery of Forcing." The Rocky Mountain Journal of Mathematics 32, no. 4 (2002): 1071-100. http://www.jstor.org/stable/44238666, 1092.

[26] Cohen, Paul J. Set Theory and the Continuum Hypothesis. Mathematics Lecture Note Series. New York: W.A. Benjamin, 1966, 151.

models in the ways we *want* to. In other words, one perspective of forcing is that although it is mathematically rigorous in its formulation, it is inherently a creative idea–one that requires some thought with regard to a desired task, rather than an objective, deduction from a set of axioms.

There are some inherent complications from a meta-theoretical standpoint in employing such a technique. The spirit of axiomatization is that the entirety of set theory is built upon the same foundation; informally, everyone has access to the same materials and rules for building proofs. Cohen's invention of forcing, although impressive and groundbreaking, runs against this spirit altogether; it's with his own tools that Cohen achieves his independence results. This is not to insinuate that forcing is an invalid or incorrect technique of *logic*–this question would be a matter of formality and validity. However, in comparison, Gödel's proof uses the notions of definability and constructibility which clearly exist within the language of set theory as provided directly by the ZF axioms. Gödel's project simply becomes proving that this new, smaller universe satisfies all the properties we want it to have. On the other hand, forcing seems to create new models which although powerful and interesting, give brand-new semantically confusing results–what does it mean for c to be bigger than any $\aleph_\tau$ if we *force* it to be? Does c escape the notion of cardinality altogether? Is it a type of infinity which is not discovered?

Another topic left ambiguous is the characterization of real numbers. For example, the mathematical intuition for a rational number is a ratio, and an integer can be described as a measure of quantity or countability. However, real numbers seem to escape any kind of intuitive description, perhaps other than the numbers which can be found on the continuum, but this simply transfers the questions onto the nature of the real line. This ambiguity is perhaps most distinctly captured when comparing the nature of the real numbers within Gödel's inner model and Cohen's forced model. In L, real numbers are constructible–there exists a countable ordinal capable of building any real number. In contrast, Cohen's approach relies on real numbers being inherently generic, so that they can be forced into the extended model. Thus, even the approaches Gödel and Cohen have for the very concept of a real number seem to diverge. On the one hand, real numbers are just ordinary numbers which can be constructed, and on the other, real numbers are objects which can be referred to without being defined. Since both CH and ~CH can be added consistently to ZF, we have no reason to prefer one characterization of the reals to the other as a matter of logic; in some sense, they are, although seemingly opposite or even contradictory, equally "correct".

Truth v. Consistency

It's useful to separate the concept of truth and consistency, since this is an area where the semantics of the English language diverge from the language of logic and set theory. Notice that both Gödel and Cohen give relative-consistency arguments, that is, (ZF + CH) is consistent if ZF itself is, and the same is true for (ZF + ~CH). The consistency of ZF, however, is not a truth available to set theory, due to Gödel's Incompleteness theorems. Thus, it's largely an open-ended question from the syntactic perspective whether these relative-consistency arguments can stand alone in set theory. One of the most interesting mathematical questions which arises from this issue is whether we have reason to add one or any of these axioms as one of the foundational axioms, that is, integrate it within the very structure of the ZF system.

Many of the arguments proposing the addition or rejection of the Continuum Hypothesis into ZF are then, by their nature, philosophical arguments. One approach is to argue that the criterion for whether a statement should be added as an axiom is dependent on their fruitfulness, that is, the results we can prove assuming the statement. However, even this judgment is not a neutral position, as both CH and $\sim CH$ have proved to offer different results for other branches of mathematics and set theory, and so choosing CH or CH simply extends the problem onto a preference for a set of mathematical results. It's hard to see how this alone can be a convincing argument to choose a side.

Gödel and Cohen both approach the independence results from a slightly more metaphysical perspective, meaning that they assume, or at least find it important, that the

Continuum Hypothesis has a natural truth value apart from a purely syntactical fruitfulness. For example, consider Gödel's construction of the $ZF + CH$ model, which uses the Axiom of Constructibility. If taken at face value, this axiom makes a rather strong semantic claim, namely, that sets are inherently constructible. Gödel himself is noted for having two polar opposite stances on the truth of the Axiom; in his original publication of the proof, he says "the proposition... added as a new axiom, seems to give a natural completion of the axioms of set theory," but six years later, reverses the position and says "[the] axiom [of constructibility] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set".[27] In other words, although Gödel uses the Axiom to construct his model and can verify all the conditions it satisfies and its syntactic relationship to ZF, he rejects the truth or plausibility of the axiom because of its conceptual implications, which clash with his subjective belief that the world of sets should be greater and more expansive and complicated than L.

Naturally then, the question arises in assessing the extent to which we can *judge* whether a proposition is true or false in mathematics given our lack of formal guidance. Gödel appeals to a certain intuitive conception of mathematics as expansive and unlimited, and it is this real characteristic to which the Axiom of Constructiblity is antithetical. Certainly, there are empirical truths like "Gödel published his proof in 1940" to which we are guided by observable facts of reality, but it's unclear what relationship mathematics has with existence and reality, since it so effortlessly transitions between the world of formal language and empirical calculations.

Additionally, Gödel holds an even stronger viewpoint that not only do theorems and axioms have an intuitive plausibility in and of themselves, but the fact that they are currently independent reveals an incompleteness or insufficiency that mathematics (and set theory) is *supposed to have*. Gödel is known for having the following reaction to the independence proof:

> "Only someone who denies that the concepts and axioms of classical set theory have any meaning could be satisfied with such a solution... For in

[27] Maddy, Penelope. "Believing the Axioms. I." The Journal of Symbolic Logic 53, no. 2 (1988): 481-511. doi:10.2307/2274520, 492

reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of reality".[28]

It was Gödel's personal opinion that the Continuum Hypothesis was false, and there was an infinity of infinities between $\aleph_0$ and c, but such a conjecture has never been proven.[29] Compare Gödel's reaction to Cohen's:

> "The ultimate response to CH must be looked at in human, almost sociological terms. We will debate, experiment, prove and conjecture until some picture emerges that satisfies this wonderful taskmaster that is our intuition".[30]

Cohen's interpretation of the independence of the Continuum Hypothesis yields to the human intuition; in essence, a central component of mathematical results is their relationship with the mathematician, and whether these concepts are *real* or describe reality accurately is in some ways not the point. It's clear that Cohen is aware of Gödel's position, but chooses to place value not on complete syntactic coherence, but rather on the human perspective, perhaps exactly what Gödel referred to when speaking about being "satisfied" with the independence results.

However, in line with Gödel's confusinginly ambivalent position is Cohen's own:

> "A point of view which the author feels may eventually come to be accepted is that CH is obviously false... It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C".[31]

It was Cohen's view that since the axioms couldn't capture the size of the continuum, it had to have a cardinality greater than $\aleph_2$.[32] This is to say, Cohen's appreciation for intuition does not prevent him from taking a stance on the truth value of the hypothesis, betraying his assumption that, like Gödel, veracity is a natural feature of mathematical statements. Notice the equivocation of "false", and the appeal to intuition in the way he says "obviously" and "unreasonable to expect"–Cohen too participates in a debate appealing to a kind of conventional characterization of the way mathematics is perceived

[28] Gödel, Kurt, and Solomon Feferman. Collected Works. Oxford: ClarendonPress, 1986, 181
[29] Wallace, 305.
[30] Cohen, Paul J. Set Theory and the Continuum Hypothesis, 1966.
[31] Ibid., 151.
[32] Cohen, Paul. "The Discovery of Forcing.", 1099.

to "naturally" behave. But since Gödel has shown that the Continuum Hypothesis can be added consistently to ZF, it seems unwarranted to say that the hypothesis is obviously false. Thus, both mathematicians ultimately hold positions influenced by their philosophical perspectives and interpretations of the independence proofs.

Conclusion

"No one shall drive us out of the paradise which Cantor has created for us."

David Hilbert, 1926

It's a strong historical clash that the two important figures in the independence proofs hold the belief that the Continuum Hypothesis is false while Cantor believed it to be true. As this paper has tried to demonstrate, a reaction to such results has a philosophical and even historic significance outside of formal language. The story of the Continuum Hypothesis can be interpreted in many ways; as a tragedy when considering how much stress Cantor felt for being unable to solve the problem; as a victory, showcasing some of the most powerful results of mathematical logic; or something in between. Mathematics doesn't provide a natural stance on how it should proceed in light of the independence results. Some have considered the question of the Continuum Hypothesis largely settled, content with its independence, or not an important consideration–and there is some truth in this position, since modern set theory continues to do important and powerful work without explicitly needing a true-or-false answer to the hypothesis. However, questions like the Continuum Problem dent mathematical history in an incredibly profound way; they ask us to reconsider the most basic ideas of truth and reality, our confidence in our intuitions and frameworks of understanding, and the relationship the human mind has with the webwork of mathematics. In this way, Hilbert is exactly right; the question that begins with Cantor still has not lost its paradisaical wonder, and never will. We are always invited to conjecture, argue, and think harder and deeper about the questions which shape the basis of mathematics as we know it.

Notes

[A]See (Ferreirós 1999) pp. 299-324 for a discussion on the emergence of the Zermelo-Fraenkel system, pp. 169-183 for the history of exchanges between Cantor and Dedekind, and pp. 382-385 for a contextual and proof summary of Gödel's consistency results. See (Devlin 1979) pp. 149-160 for a complete walkthrough of the Axiom of Constructability, and differences between V and L. See (Wolf 2005) pp. 91-94 for a description of the von Neumann hierarchy and pp. 225-242 for a different discussion on the differences between V and L, the concept and definition of relativization, and the important theorems necessary to show the consistency of $ZF + (V = L)$. See (Bagaria 2019) for a general summary of the main points and ideas behind both Gödel and Cohen's proof.

[B]See (Wolf 2005) pp. 225-242 for an introduction to forcing and a summary of Cohen's proof. See (Cohen 1966) for the original explanation of forcing by Cohen. This source is noticeably the most technical of the sources used, although Cohen is careful to introduce the context, goals, and intuitions before each chapter. See pp. 85-106 for Cohen's interpretation of Gödel's proof, and pp. 107-136 for Cohen's tour of the forcing concepts and proof of independence, that is, the building of the extended model. See (Cohen 2002) for Cohen's own perspective on his proof, Gödel's proof, and the independence results. Unlike (Cohen 1966), this paper is less technical and focuses on the driving intuition behind forcing, as well as a reflection on the philosophy of mathematics.

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